বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2022

(Under CBCS Pattern)

Semester - IV

Subject : MATHEMATICS

Paper : C 8 - T

Riemann Integration and Series of Functions

Full Marks : 60

Time : 3 Hours

Candidates are required to give their answers in their own words as far as practicable.
The figures in the margin indicate full marks.

1. Answer any ***five*** questions : 2×5=10

(a) Let $f:[a,b]\to R$ be a bounded function and P be any partition over $[a, b]$. Define lower sum $L(P,f)$ and upper sum $U(P,f)$.

(b) Let $f:[a,b]\to R$ be integrable on $[a, b]$. If M and m be respectively the supremum and infimum of f on $[a, b]$, prove that $m(b-a)\le\int_a^b f dx\le M(b-a)$.

(c) Prove or disprove : if f is differentiable on [0, 1], the relation $\int_0^1 f' dx = f(1)-f(0)$ is not always true.

(d) A function f is continuous in the interval $[a, \infty)$ and $f(x) \to A(\neq 0)$ as $x \to \infty$. Can the integral $\int_a^\infty f(x)\,dx$ converge?

(e) Discuss the convergence of $\int_0^1 e^{-x}.x^{n-1}dx$.

(f) Give examples of (i) everywhere convergent power series (ii) nowhere convergent power series.

(g) Let D be a finite subset of R. If a sequence of real valued functions $\{f_n(x)\}_n$ on D converges pointwise to $f(x)$, then show that it also converges uniformly to $f(x)$.

(h) Let $\sum_n f_n(x)$ be a series of functions defined on $D(\subset R)$. Explain when this series is said to be uniformly convergent on D.

2. Answer any ***four*** questions : 5×4=20

(a) Find the Fourier series of the periodic function f with period 2π, where $f(x) = \begin{cases} 0, -\pi < x < a \\ 1, \ \ a \le x \le b \\ 0, \ \ b < x < \pi \end{cases}$. Find the sum of the series at $x = 4\pi + a$ and deduce that $\sum_{n=1}^{\infty} \frac{\sin n(b-a)}{n} = \frac{\pi-(b-a)}{2}$.

(b) Evaluate $\int_2^5 (x^2 - x)\,dx$ by using the geometric partition of [2, 5] into n subintervals.

(c) Find the radius of convergence of the power series $\sum_{n=1}^{\infty} (-1)^{n-1}.\frac{x^n}{n}$ and discuss its convergence at each end of the interval.

(d) Show that $\sum_{n=0}^{\infty} x^n$ uniformly on $[-a, a]$ where $0 < a < 1$, but $\sum_{n=1}^{\infty} \left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2 x^2}\right]$ is not uniformly convergent on R.

(e) Show that $\int_1^\infty x^{m-1}(\log x)^n dx$ is convergent if and only if $m<0,\ n>-1$.

(f) Let f be a continuous function on R and define $F(x)=\int_{x-1}^{x+1} f(t)\,dt,\ x\in R$. Show that F is differentiable on R and compute F'.

3. Answer any ***three*** questions : 10×3=30

(a) (i) State and prove the fundamental theorem of integral calculus.

(ii) If $0\le x\le 1$ then show that $\frac{x^2}{\sqrt{2}}\le\frac{x^2}{\sqrt{1+x}}\le x^2$ and hence show that $\frac{1}{3\sqrt{2}}\le\int_0^1\frac{x^2}{\sqrt{1+x}}\le\frac{1}{3}$. 5+5

(b) (i) If f is a piecewise continuous function or a bounded piecewise monotonic function on $[a, b]$, then f is R —integrable over $[a, b]$. 3+3

(ii) Show that $\int_\pi^\infty \frac{\sin x}{x}dx$ converges but not absolutely. 4

(c) (i) Let $\sum_n u_n(x)$ be a series of real valued function defined on $[a, b]$ and each $u_n(x)$ is R—integrable on $[a, b]$. If the series converges uniformly to f on $[a, b]$, then prove that f is R —integrable on $[a, b]$ and

$$\int_a^b\left[\sum_{n=1}^\infty u_n(x)\right]dx=\sum_{n=1}^\infty\int_a^b u_n(x)\,dx.$$

Give an example to show that the condition of uniforms convergence of $\sum_n u_n(x)$ is only a sufficient condition but not necessary. 5+2

(ii) Find the region of convergence of the series $\sum_{n=1}^\infty \frac{x^{3n}}{2^n}$. 3

(d) (i) Verify that the function $y = x^3 \sin\frac{1}{x}$ for $x \neq 0$ and $y = 0$ for $x = 0$ in the interval $[-\pi, \pi]$ is continuous together with its first derivative but does not satisfy the conditions of Dirchlet's theorem. Can it be expanded into a Fourier series in the interval $[-\pi, \pi]$. 5

(ii) Prove that the integral $\int_0^{\frac{\pi}{2}} \sin x \log \sin x \, dx$ exists and find its value. 5

(e) (i) Let $f_n(x) = |x|^{1+\frac{1}{n}}, x \in [-1,1]$. Show that $\{f_n\}_n$ is uniformly convergent on [–1, 1]. Also show that each f_n is differentiable on [–1, 1] but the limit function is not differentiable for all x in [–1, 1]. 2+2+2

(ii) Prove or disprove : $\{\tan^{-1} nx\}_n$ is not uniformly convergent on any interval which includes zero. 4